Tutorial 3 (Jan 29, 31)

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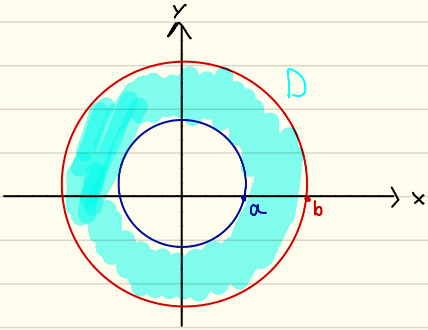
Q1) Find the average value of the function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ over the annular region $D = \{(x,y) \in \mathbb{R}^2 \mid a^2 \leq x^2+y^2 \leq b^2\}$, where $0 < a < b$.

Sol) By definition, average of f over $D = \frac{1}{\text{Area}(D)} \iint_D f(x,y)\,dA$

Picture:



Step 1: Express D in terms of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

$$D = \{(r,\theta) \in (0,+\infty) \times [-\pi,\pi) \mid a \leq r \leq b\}$$

Step 2: Compute Area(D).

Method 1: By double integral: $\text{Area}(D) = \iint_D 1\,dA$

$$= \int_{-\pi}^{\pi}\int_a^b r\,dr\,d\theta = \int_{-\pi}^{\pi}\left[\frac{r^2}{2}\right]_a^b d\theta = \int_{-\pi}^{\pi}\left(\frac{b^2-a^2}{2}\right)d\theta = \left[\frac{b^2-a^2}{2}\theta\right]_{-\pi}^{\pi} = \pi(b^2-a^2)$$

Method 2: By elementary geometry: Area(D) = Area(O) − Area(o)

$$= \pi b^2 - \pi a^2 = \pi(b^2-a^2)$$

Step 3 : Compute $\iint_D f(x,y)\,dA$ using polar coordinates.

then $f(r,\theta) = \frac{1}{\sqrt{r^2}} = \frac{1}{r}$

$$\therefore \iint_D f(x,y)\,dA = \int_{-\pi}^{\pi}\int_a^b \frac{1}{r}\cdot r\,dr\,d\theta$$

$$= \left(\int_{-\pi}^{\pi} d\theta\right)\cdot\left(\int_a^b dr\right)$$

$$= 2\pi\cdot(b-a)$$

Step 4 : Compute the average of f over D.

$$\text{Average} = \frac{1}{\text{Area}(D)}\iint_D f(x,y)\,dA = \frac{1}{\pi(b^2-a^2)}\cdot 2\pi(b-a) = \frac{2}{b+a}$$

Q2) Evaluate $\underbrace{\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx}_{\mathrm{I}} + \underbrace{\int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx}_{\mathrm{II}} + \underbrace{\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx}_{\mathrm{III}}$.

Sol) Method 1: Direct computation.

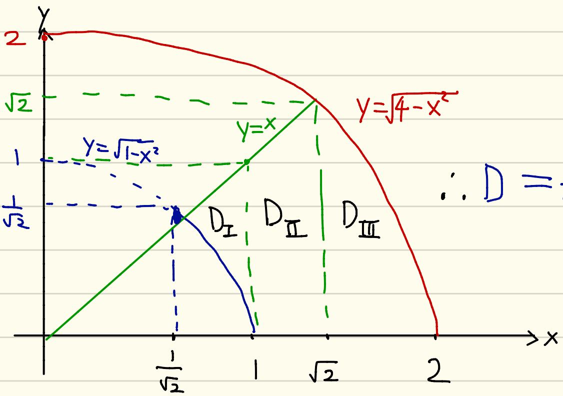
$$\mathrm{I}+\mathrm{II}+\mathrm{III} = \int_{\frac{1}{\sqrt{2}}}^{1}\left[\frac{xy^2}{2}\right]_{\sqrt{1-x^2}}^{x} dx + \int_{1}^{\sqrt{2}}\left[\frac{xy^2}{2}\right]_{0}^{x} dx + \int_{\sqrt{2}}^{2}\left[\frac{xy^2}{2}\right]_{0}^{\sqrt{4-x^2}} dx$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1}\left(\frac{x^3}{2} - \frac{x(1-x^2)}{2}\right)dx + \int_{1}^{\sqrt{2}}\left[\frac{x^3}{2}\right]dx + \int_{\sqrt{2}}^{2}\frac{x(4-x^2)}{2}dx = \cdots = \frac{15}{16}$$

Method 2: Understand the domain of integration $D := D_{\mathrm{I}} \cup D_{\mathrm{II}} \cup D_{\mathrm{III}}$.

where
$$\begin{cases} D_{\mathrm{I}} = \{(x,y)\in\mathbb{R}^2 \mid \frac{1}{\sqrt{2}} \le x \le 1,\ \sqrt{1-x^2} \le y \le x\} \\ D_{\mathrm{II}} = \{(x,y)\in\mathbb{R}^2 \mid 1 \le x \le \sqrt{2},\ 0 \le y \le x\} \\ D_{\mathrm{III}} = \{(x,y)\in\mathbb{R}^2 \mid \sqrt{2} \le x \le 2,\ 0 \le y \le \sqrt{4-x^2}\} \end{cases}$$

Picture:



$\therefore D = \{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{4};\ 1 \le r \le 2\}$

$$\therefore \quad \mathrm{I} + \mathrm{II} + \mathrm{III} = \iint_D xy\, dA = \int_0^{\frac{\pi}{4}} \int_1^2 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$$

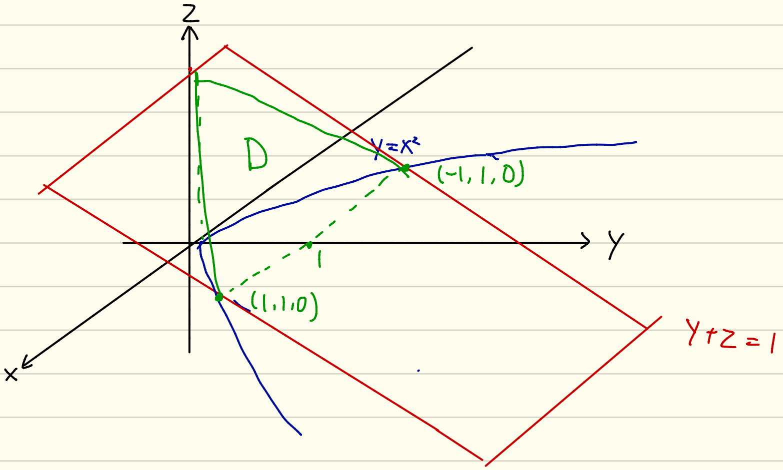
$$= \left(\int_0^{\frac{\pi}{4}} \cos\theta \sin\theta \, d\theta \right) \cdot \left(\int_1^2 r^3\, dr \right)$$

$$= \left[\frac{\sin^2\theta}{2} \right]_0^{\frac{\pi}{4}} \cdot \left[\frac{r^4}{4} \right]_1^2$$

$$= \frac{1}{4} \cdot \frac{15}{4} = \frac{15}{16}$$

Q3) Find the volume of the solid enclosed by the cylinder $y=x^2$ and the planes $z=0$, $y+z=1$.

Sol) Step 1: Sketch the solid D.



Step 2: Express D in terms of coordinates.

$$D=\{(x,y,z)\in\mathbb{R}^3 \mid -1\leq x\leq 1,\ x^2\leq y\leq 1,\ 0\leq z\leq 1-y\}$$

Step 3: Compute the volume of D by triple integral.

$$\text{Volume} = \iiint_D 1 \cdot dV = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} [z]_0^{1-y}\,dy\,dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} (1-y)\,dy\,dx$$

$$= \int_{-1}^{1} \left[y - \frac{y^2}{2}\right]_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \left(\left(1-\frac{1}{2}\right) - \left(x^2 - \frac{x^4}{2}\right)\right) dx$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx$$

$$= 2\int_0^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx$$

$$= 2\left[\frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right)$$

$$= \frac{8}{15}$$

Remark: Alternatively, the volume can be computed by a double integral

$$\iint_{D'} f(x,y)\,dA, \text{ where } \begin{cases} D' = \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 1,\ x^2 \le y \le 1\} \\ f(x,y) = 1-y \end{cases}$$